Singularities of Secant Varieties

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Singularities of secant varieties: what is known

Introduction to secant varieties

We first describe the basic set-up.

- Z is a smooth projective variety of dimension n.
- L is a very ample line bundle inducing the embedding

  $$Z \xhookrightarrow{|L|} \mathbb{P}^N = \mathbb{P}^{h^0(L)-1}$$

  by its complete linear series.
- Define the **secant variety** as

  $$\Sigma := \overline{\bigcup_{\substack{x_1, x_2 \in Z \\ x_1 \neq x_2}} (\text{Line joining } x_1 \text{ and } x_2)}.$$

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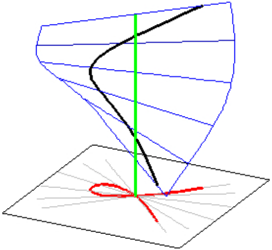
Facts

Σ is irreducible of dimension $\leq n+n+1 = 2n+1$.

Why do we care about secant varieties?

Arises from classical construction of projection!

Question: Given a smooth curve $C \subset \mathbb{P}^N$, can we find $p \in \mathbb{P}^N \backslash \{C\}$ such that the projection produces isomorphic image?



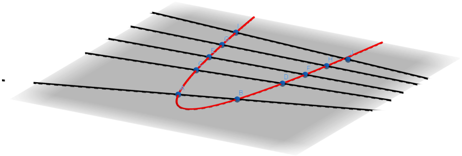
Upshot: Projection from $p \in \mathbb{P}^3 \backslash \{C\}$ on secant lines produces singularities!

Leads to the Whitney embedding theorem.

Introduction to secant varieties

Examples of defective secant varieties.

(1) $Z = \mathbb{P}^1 \xhookrightarrow{|\mathcal{O}_{\mathbb{P}^1}(2)|} \mathbb{P}^2$, $[z_0, z_1] \mapsto [z_0^2, z_0 z_1, z_1^2]$ (Plane conic).



In this case, $\Sigma = \mathbb{P}^2$.

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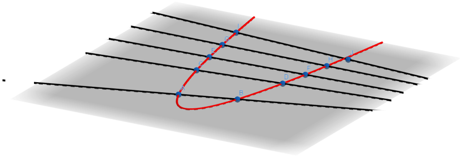


In this case, $\Sigma = \mathbb{P}^2$.

(2) (Second Veronese surface) Consider $Z = \mathbb{P}^2 \xhookrightarrow{|\mathcal{O}_{\mathbb{P}^2}(2)|} \mathbb{P}^5$ given by

$$[z_0, z_1, z_2] \mapsto [z_0^2, z_1^2, z_2^2, z_0 z_1, z_1 z_2, z_0 z_2].$$

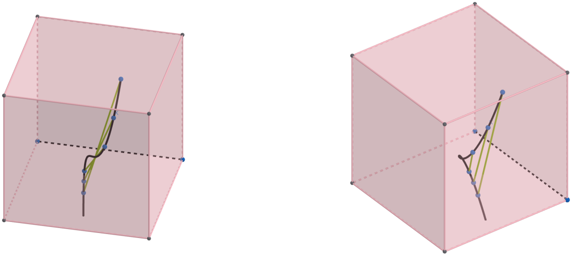
One can show that in this case $\dim \Sigma = 4$. It is given by

$$\left\{ \det \left(\begin{bmatrix} X_0 & X_3 & X_4 \\ X_3 & X_1 & X_5 \\ X_4 & X_5 & X_2 \end{bmatrix} \right) = 0 \right\}.$$

Introduction to secant varieties

Example of non-defective secant variety.

(1) $Z = \mathbb{P}^1 \xhookrightarrow{|\mathcal{O}_{\mathbb{P}^1}(3)|} \mathbb{P}^3$, $[z_0, z_1] \mapsto [z_0^3, z_0^2 z_1, z_0 z_1^2, z_1^3]$ (Twisted cubic).



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Introduction to secant varieties

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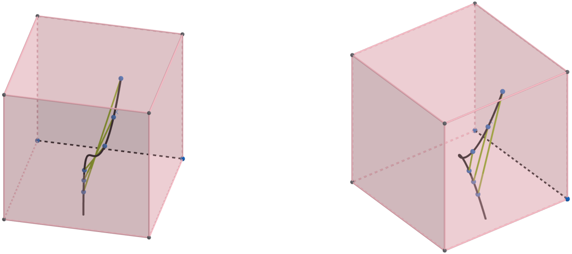

In this case, $\Sigma = \mathbb{P}^3$.

(2) Consider $Z = \mathbb{P}^1 \xhookrightarrow{|\mathcal{O}_{\mathbb{P}^1}(4)|} \mathbb{P}^4$. One can show that in this case $\dim \Sigma = 3$. It is given by

$$\left\{ \det \left(\begin{bmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{bmatrix} \right) = 0 \right\}.$$

Singularities of secant varieties

Facts: If the embedding of the smooth variety $Z \subset \mathbb{P}^N$ is 3-very ample, then:

- Σ has the expected dimension $2\dim Z + 1$.
- By construction $Z \subset \Sigma$. If $\Sigma \neq \mathbb{P}^N$ then $\Sigma_{\text{sing}} = Z$.

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Theorem

If $Z \subset \mathbb{P}^N$ is sufficiently positive *(much more beyond 3-very ampleness)*, then:

- (Ullery, '18) Σ is normal
- (Chou–Song, '18) Σ has Du Bois singularities.
- (Chou–Song, '18) Σ has rational singularities $\iff H^i(\mathcal{O}_Z) = 0$ for $i > 0$.
- (Chou–Song, '18) Σ is Cohen-Macaulay $\iff H^i(\mathcal{O}_Z) = 0$ for $0 < i < \dim Z$.

Description of Du Bois complex of secant varieties: higher singularities

Hodge decomposition of singular cohomology

X is a projective variety of dimension n.

- For $k \in \mathbb{Z}$, the **singular cohomology** $H^k(X, \mathbb{C})$ is a topological object.
- For **smooth** X, we have

$$\boxed{H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(\Omega_X^p)}.$$

Ω_X^p is the **p-th Kähler differential**.

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Upshot: Topology (singular coh) $\overset{\text{Hodge th}}{\longleftrightarrow}$ Algebra/Analysis (sheaf coh).

Hodge decomposition of singular cohomology

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- For **arbitrary** X, we have

$$\boxed{H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(\underline{\Omega}_X^p)}.$$

In the above, $\underline{\Omega}_X^p \in D^b(\mathrm{Coh}(X))$ is the **p-th Du Bois complex of X** (constructed by Du Bois '81). ▸ Example

Upshot: Topology (sing coh) $\xleftrightarrow{\text{Hodge th}}$ Algebra/Analysis (coh of complex).

(Higher) Du Bois singularities

For any X of dimension n and for any p, we have the objects

Ω_X^p	$\underline{\Omega}_X^p$
***p*-th Kähler** Computes singular cohomology for smooth projective X	***p*-th Du Bois complex** Computes singular cohomology for arbitrary projective X

They are the same (i.e. $\Omega_X^p \to \underline{\Omega}_X^p$ are quasi-isomorphisms) for every p when X is **smooth**.

The idea is to compare them to extract singularity information.

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X has **Du Bois singularities** if $\mathcal{O}_X = \Omega^0_X \xrightarrow{\cong} \underline{\Omega}^0_X$.

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Definition (Mustaţă–Olano–Popa–Witaszek, Jung–Kim–Saito–Yoon ’22/23)

Assume X is **LCI** and $k \geq 1$. X is said to have **k-Du Bois singularities** if

$$\Omega^p_X \xrightarrow{\cong} \underline{\Omega}^p_X \text{ for all } p = 0, \cdots, k. \qquad (1)$$

- **Example:** $\left\{x_1^{d_1} + \cdots + x_n^{d_n} = 0\right\}$ is k-Du Bois $\iff \sum \frac{1}{d_i} \geq k + 1$.

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- **Example:** $\left\{x_1^{d_1} + \cdots + x_n^{d_n} = 0\right\}$ is k-Du Bois $\iff \sum \frac{1}{d_i} \geq k+1$.
- No example of non-LCI variety X when (1) holds for $k \geq 1$.

(Higher) rational singularities

For any X of dimension n and for any p, we have the objects

Ω_X^p	$\underline{\Omega}_X^p$	$\mathbb{D}_X(\underline{\Omega}_X^{n-p})$
p-th Kähler computes sing. coh. for sm. proj. X	**p-th Du Bois complex** computes sing. coh. for any projective X	**Dual of $(n-p)$-th Du Bois complex**

All are same (i.e., $\Omega_X^p \to \underline{\Omega}_X^p \to \mathbb{D}_X(\underline{\Omega}_X^{n-p})$ are quasi-iso) when X is **smooth**.

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Definition (non–standard)

X has **rational singularities** if $\mathcal{O}_X = \Omega_X^0 \xrightarrow{\cong} \mathbb{D}_X(\underline{\Omega}_X^n)$.

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X has **rational singularities** if $\mathcal{O}_X = \Omega_X^0 \xrightarrow{\cong} \mathbb{D}_X(\underline{\Omega}_X^n)$.

Definition (Friedman–Laza '22)

Assume X is **LCI** and $k \geq 1$. X has ***k*-rational singularities** if

$$\Omega_X^p \xrightarrow{\cong} \mathbb{D}_X(\underline{\Omega}_X^{n-p}) \ \text{ for all } \ p = 0, \cdots, k. \tag{2}$$

Example: $\left\{x_1^{d_1} + \cdots + x_n^{d_n} = 0\right\}$ is k-rational $\iff \sum \frac{1}{d_i} > k+1$.

Du Bois complex of secant varieties

$$\begin{tikzcd}
\underline{\Omega}_X^p & & \mathbb{D}_X(\underline{\Omega}_X^{n-p}) \\
& \Omega_X^p \arrow[ul, leftrightarrow, "\text{higher DB}"] \arrow[ur, leftrightarrow, "\text{higher rational}"'] &
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Theorem (Chou–Song '18)

Assume $Z \subset \mathbb{P}^N$ is sufficiently positive. Then:
(1) $\Omega^{[0]}_\Sigma \cong \underline{\Omega}^0_\Sigma$.
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k-Hodge Rational Homology Manifolds

Introduction to HRH-level

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Dual of $(n-p)$-th Du Bois complex

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\end{tikzcd}$$

Definition (Dirks–Olano–R '25)

X is a variety of dimension n. We say that X is a **k-Hodge rational homology "manifold"** (abbreviated as $\mathrm{HRH}(X) \geq k$) if

$$\underline{\Omega}_X^p \cong \mathbb{D}_X(\underline{\Omega}_X^{n-p}) \text{ for all } p = 0, \cdots, k.$$

Remark: Many non-LCI examples that have non-trivial HRH level!

Connection to Rational homology manifolds

A variety X of dimension n is called **rational homology "manifold" (RHM)** if

$$H^i_{\{x\}}(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 2n \\ 0 & i < 2n \end{cases} \text{ for all } x \in X.$$

Remark: Widely studied notion in topology!

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The (complexified) local cohomology spaces are equipped with Hodge filtrations:

$$\cdots \subseteq F_{-n}H^i_{\{x\}}(X,\mathbb{C}) \subseteq F_{-n+1}H^i_{\{x\}}(X,\mathbb{C}) \subseteq \cdots$$

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Theorem (Dirks–Olano–R '25)

Let X be a variety of dimension n. Then

$$\mathrm{HRH}(X) \geq k \iff F_{k-n}H^i_{\{x\}}(X,\mathbb{C}) = \begin{cases} \mathbb{C} & i = 2n \\ 0 & i < 2n \end{cases} \text{ for all } x \in X.$$

X is RHM $\iff$ $\mathrm{HRH}(X) \geq k$ for all k.

Usefulness: Partial Poincaré duality

For projective rational homology manifolds X of dimension n, we have

$$\textbf{Poincaré duality: } H^{n-i}(X,\mathbb{C}) \cong H^{n+i}(X,\mathbb{C})^*.$$

Theorem (Dirks–Olano–R '25)

Let X be a projective variety of dimension n with $\mathrm{HRH}(X) \geq k$. Then

$$F_{k-n}H^{n-i}(X,\mathbb{C}) \cong F_kH^{n+i}(X,\mathbb{C})^*.$$

In other words, it satisfies Poincaré duality up to Hodge level k.

HRH levels of secant varieties

Theorem (Chen–Dirks–Olano–R '26)

Assume $Z \subset \mathbb{P}^N$ is a 3-very ample embedding of a smooth projective variety Z. Then

$$\mathrm{HRH}(\Sigma) = \begin{cases} \infty & Z \cong \mathbb{P}^1, \\ 0 & H^i(\mathcal{O}_Z) = 0 \ \forall\ i \geq 1, \\ -1 & \text{otherwise.} \end{cases}$$

Local cohomological invariants

Connection to local cohomology

Let $X \hookrightarrow W$ be a closed embedding of a c-codimensional variety X inside a smooth variety. The local cohomology modules are defined as

$$\mathcal{H}^q_X(\mathcal{O}_W) = \varinjlim \mathcal{E}xt^q_{\mathcal{O}_W}(\mathcal{O}_W/\mathcal{J}^{k+1}_{X/W}, \mathcal{O}_W)$$

We know: $\min \{q \mid \mathcal{H}^q_X(\mathcal{O}_W) \neq 0\} = c$.

We have the local cohomology modules

$\mathcal{H}^c_X(\mathcal{O}_W)$	$\mathcal{H}^{c+1}_X(\mathcal{O}_W)$	$\mathcal{H}^{c+2}_X(\mathcal{O}_W)$	$\cdots$

The **local cohomological defect** is independent of the embedding (Popa–Shen '24):

$$\operatorname{lcdef}(X) := \max \left\{k \mid \mathcal{H}^{c+k}_X(\mathcal{O}_W) \neq 0\right\}.$$

X is **Cohomological complete intersection (CCI)** if $\mathcal{H}^j_X(\mathcal{O}_W) = 0$ for $j > c$, i.e., lcdef(X) $= 0$, or the above looks like

$\mathcal{H}^c_X(\mathcal{O}_W)$	0	0	$\cdots$

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$$\operatorname{lcdef}(\Sigma) = \begin{cases} n-1 & n \geq 2, H^1(\mathcal{O}_Z) \neq 0, \\ n-2 & n \geq 2, H^1(\mathcal{O}_Z) = 0, \\ 0 & \text{otherwise (i.e., } n = 1). \end{cases}$$

The invariant $c(X)$

Let $X \hookrightarrow W$ be a closed embedding of a c-codimensional variety X inside a smooth variety W of dimension n.

The local cohomology modules are equipped with Hodge filtration:

$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_{-n+2}\mathcal{H}^{c}_{X}(\mathcal{O}_W)$	$F_{-n+2}\mathcal{H}^{c+1}_{X}(\mathcal{O}_W)$	$F_{-n+2}\mathcal{H}^{c+2}_{X}(\mathcal{O}_W)$	$\cdots$
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Definition (Chen–Dirks–Olano ’25)

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$F_{-n}\mathcal{H}^c_X(\mathcal{O}_W)$	0	0	$\cdots$

$$c(X) \geq 0$$

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$F_{-n}\mathcal{H}^c_X(\mathcal{O}_W)$	0	0	$\cdots$

$$c(X) \geq 1$$

Definition (Chen–Dirks–Olano ’25)

$$c(X) := \max\left\{k \mid F_{p-n}\mathcal{H}^p_X(\mathcal{O}_W) = 0\,\forall\, p \leq k\right\}.$$

Diagram of implications

X is any variety.

$$\begin{tikzcd}
\text{Rational} \arrow[r, Rightarrow, "\text{[DOR]}"] \arrow[d, Rightarrow] & \mathrm{HRH}(X) \geq 0 \arrow[d, Rightarrow, "\text{[DOR]}"] \arrow[l, Rightarrow, bend right, "\text{+Du Bois}"'] \\
\text{Cohen-Macaulay} \arrow[r, Rightarrow, "\text{[CDO]}"] & c(X) \geq 0 \arrow[l, Rightarrow, bend left, "\text{+Du Bois}"]
\end{tikzcd}$$

Refinement of Chou-Song's theorem

$$\begin{tikzcd}
\text{Rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{HRH}(X) \geq 0 \arrow[d, Rightarrow] \arrow[l, Rightarrow, bend right, "+\text{Du Bois}"'] \\
\text{Cohen-Macaulay} \arrow[r, Rightarrow] & c(X) \geq 0 \arrow[l, Rightarrow, bend left, "+\text{Du Bois}"]
\end{tikzcd}$$

Theorem (Chou–Song '18)

Assume $Z \subset \mathbb{P}^N$ is sufficiently positive. Then:

- Σ has Du Bois singularities.
- Σ has rational singularities $\iff H^i(\mathcal{O}_Z) = 0$ for $i > 0$.
- Σ is Cohen-Macaulay $\iff H^i(\mathcal{O}_Z) = 0$ for $0 < i < \dim Z$.

Theorem (Chen–Dirks–Olano–R '26)

Assume $Z \subset \mathbb{P}^N$ is 3-very ample. Then:

- $\mathrm{HRH}(\Sigma) \geq 0 \iff H^i(\mathcal{O}_Z) = 0$ for $i > 0$.
- $c(\Sigma) \geq 0 \iff H^i(\mathcal{O}_Z) = 0$ for $0 < i < \dim Z$.

Classification

We also compute the singular and intersection cohomology of Σ.

Corollary (Chen–Dirks–Olano–R '26)

Assume $Z \subset \mathbb{P}^N$ is 3-very ample. Then the following are equivalent:

- Σ is a rational homology manifold.
- Σ has quotient singularities.
- Σ has $\mathbb{Q}$-factorial singularities.
- $Z \cong \mathbb{P}^1$.

Example: $\Sigma := \Sigma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ with $d \geq 3$.

(1) $\mathrm{lcdef}(\Sigma) = 0$, $c(\Sigma) = \infty$, $\mathrm{HRH}(\Sigma) = 0$.
(2) *Intersection Hodge numbers.* By Poincaré duality, we only need to compute $\mathrm{IH}^j(\Sigma)$ for $0 \leq j \leq 5$, which are given by:

j	0	1	2	3	4	5
$\mathrm{IH}^j(\Sigma)$	$\mathbb{Q}^H$	0	$\mathbb{Q}^H(-1)^{\oplus 2}$	0	$\mathbb{Q}^H(-2)^{\oplus 2}$	0

The intersection Hodge numbers $\mathrm{I}\underline{\mathrm{h}}^{p,q}(\Sigma) := \dim \mathrm{Gr}_F^p \mathrm{IH}^{p+q}(\Sigma)$ are given by

$$\mathrm{I}\underline{\mathrm{h}}^{p,q}(\Sigma) = \begin{cases} 1 & (p,q) \in \{(0,0),(5,5)\} \\ 2 & (p,q) \in \{(1,1),(2,2),(3,3),(4,4)\} \\ 0 & \text{otherwise} \end{cases} .$$

Example: $\Sigma := \Sigma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ with $d \geq 3$.

(3) *Hodge-Du Bois numbers.* $H^j(\Sigma)$ is pure of weight j, and are given by

j	0	1	2	3	4	5	6	7	8	9	10
$H^j(\Sigma)$	$\mathbb{Q}^H$	0	$\mathbb{Q}^H(-1)$	0	$\mathbb{Q}^H(-2)$	0	$\mathbb{Q}^H(-3)^{\oplus 2}$	0	$\mathbb{Q}^H(-4)^{\oplus 2}$	0	$\mathbb{Q}^H(-5)$

Hodge-Du Bois numbers $\underline{h}^{p,q}(\Sigma) := \dim \operatorname{Gr}_F^p H^{p+q}(\Sigma)$ are given by

$$\underline{h}^{p,q}(\Sigma) = \begin{cases} 1 & (p,q) \in \{(0,0),(1,1),(2,2),(5,5)\} \\ 2 & (p,q) \in \{(3,3),(4,4)\} \\ 0 & \text{otherwise} \end{cases} .$$

(4) *Defect in $\mathbb{Q}$-factoriality.* $\sigma(\Sigma) = 1,$ and $\sigma^{\mathrm{an}}(\Sigma; y) = 1$ for any $y \in \mathbb{P}^2$.

Thank you!

Some computations of DB complexes

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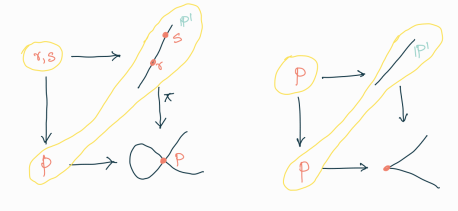

Node

$$\underline{\Omega}^0_X$$
$$= [\mathcal{O}_p \oplus \pi_*\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{r,s}]$$
$$= \mathcal{O}_X$$

$$\underline{\Omega}^1_X$$
$$= [\Omega^1_p \oplus \pi_*\Omega^1_{\mathbb{P}^1} \to \Omega^1_{r,s}]$$
$$= \pi_*\Omega^1_{\mathbb{P}^1} \neq \Omega^1_X$$

Cusp

$$\underline{\Omega}^0_X$$
$$= [\mathcal{O}_p \oplus \pi_*\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_p]$$
$$= \pi_*\mathcal{O}_{\mathbb{P}^1} \neq \mathcal{O}_X$$

$$\underline{\Omega}^1_X$$
$$= [\Omega^1_p \oplus \pi_*\Omega^1_{\mathbb{P}^1} \to \Omega^1_p]$$
$$= \pi_*\Omega^1_{\mathbb{P}^1} \neq \Omega^1_X$$